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ON MULTIPLE SOLUTIONS OF SINGULARLY PERTURBED SYSTEMS IN THE CO--ETC(U)

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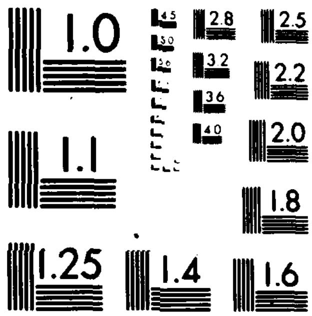

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ON MULTIPLE SOLUTIONS OF SINGULARLY PERTURBED SYSTEMS IN THE CONDITIONALLY STABLE CASE

ROBERT E. O'MALLEY, JR.

1. INTRODUCTION.

Let us consider systems

$$\dot{x} = f(x,y,t,\varepsilon) \ , \qquad \varepsilon\dot{y} = g(x,y,t,\varepsilon) \ , \tag{1}$$

of m + n ordinary differential equations on a finite interval, say $0 \le t \le 1$, subject to q initial conditions

$$A(x(0),y(0),\varepsilon) = 0 \ , \tag{2}$$

and r terminal conditions

$$B(x(1),y(1),\varepsilon) = 0 \ , \tag{3}$$

with $q + r = m + n$. We shall assume that f, g, A, and B have asymptotic series expansions in ε with coefficients being smooth functions of the remaining variables, and we shall seek the asymptotic behavior of solutions under the condition that the $n \times n$ Jacobian matrix $g_y(x,y,t,0)$ has a hyperbolic splitting with $k > 0$ (strictly) stable and $n - k > 0$ (strictly) unstable eigenvalues for all x and y and for $0 \le t \le 1$. We shall also suppose that $q \ge k$ and $r \ge n - k$, since linear examples suggest that a limiting solution as the small positive parameter ε tends to zero is unlikely to occur otherwise.

The reader should realize that the corresponding initial value problem with $k = n$ has a well-understood asymptotic solution, as presented in Wasow (1965) and O'Malley (1974). Initial value problems with a fixed number of purely imaginary

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper seeks the asymptotic behavior of solutions to nonlinear singularly perturbed boundary value problems for ordinary differential equations. Presuming a jacobian matrix has a fixed number of (strictly) stable and unstable eigenvalues, the limiting solution can often be obtained from a reduced problem. Boundary layer behavior will satisfy a conditionally stable system, and multiple solutions will often result. The asymptotic structure of solutions is helpful in developing numerical solution schemes, and is vital in various applications (including some optimal control problems).

eigenvalues have been considered by Hoppensteadt and Miranker (1976) and Kreiss (1979), while Vasil'eva and Butuzov (1978) and O'Malley and Flaherty (1980) discuss problems for which g_y has a nullspace. We note that such problems can be considerably more complicated when eigenvalues of g_y cross or approach the imaginary axis (cf., e.g., the resonance examples of Ackerberg and O'Malley (1970) and the initial value problem of Levinson (1949)). Finally, note that the strict eigenvalue stability assumptions can be weakened in "boundary layer regions" (cf. Howes and O'Malley (1980)). The two-point problems arise naturally in optimal control theory (cf. Kokotovic et al. (1976) and O'Malley (1978)), among many other applications. Moreover, knowing about the asymptotic behavior of solutions is extremely helpful in developing schemes for the numerical solution of stiff boundary value problems (cf. Hemker and Miller (1979) and Flaherty and O'Malley (1980)).

2. THE ASYMPTOTIC APPROXIMATIONS.

With the assumed hyperbolic splitting, we must expect solutions to feature nonuniform convergence as $\varepsilon \to 0$ (i.e., boundary layers) near both endpoints. Indeed, it is natural to seek bounded (uniform) asymptotic solutions in the form

$$\begin{aligned} x(t,\varepsilon) &= X(t,\varepsilon) + \varepsilon\xi(\tau,\varepsilon) + \varepsilon\eta(\sigma,\varepsilon) , \\ y(t,\varepsilon) &= Y(t,\varepsilon) + \mu(\tau,\varepsilon) + \nu(\sigma,\varepsilon) , \end{aligned} \tag{4}$$

on $0 \le t \le 1$, where the outer solution $(X(t,\varepsilon),Y(t,\varepsilon))$ represents the solution asymptotically within $(0,1)$, where the initial layer correction $(\varepsilon\xi(\tau,\varepsilon),\mu(\tau,\varepsilon))$ decays to zero exponentially as the stretched variable

$$\tau = t/\varepsilon \tag{5}$$

tends to infinity, and where the terminal layer correction $(\varepsilon\eta(\sigma,\varepsilon),\nu(\sigma,\varepsilon))$ goes to zero as

$$\sigma = (1 - t)/\varepsilon \tag{6}$$

becomes infinite. Within $(0,1)$, then, such solutions are represented by an outer expansion

$$\begin{pmatrix} X(t,\varepsilon) \\ Y(t,\varepsilon) \end{pmatrix} \sim \sum_{j=0}^{\infty} \begin{pmatrix} X_j(t) \\ Y_j(t) \end{pmatrix} \varepsilon^j . \tag{7}$$

The limiting uniform approximation corresponding to (4) is

$$x(t,\varepsilon) = X_0(t) + O(\varepsilon) ,$$

$$y(t,\varepsilon) = Y_0(t) + \mu_0(\tau) + \nu_0(\sigma) + O(\varepsilon) ,$$

on $0 \le t \le 1$. At $t = 0$, the singularly perturbed or fast vector y usually has a discontinuous limit, jumping from $y(0,0) = Y_0(0) + \mu_0(0)$ to $Y_0(0)$ at $t = 0^+$. An analogous Heaviside discontinuity occurs near $t = 1$ whenever $\mu_0(0) \neq 0$, and the derivative $\dot{y}(t,\varepsilon)$ generally features delta-function type impulses as $\varepsilon \to 0$ at both endpoints. (The relation of such observations to linear systems theory is of considerable current interest (cf. Francis (1979) and Verghese (1978)). For problems linear in the fast variable y, we can also find unbounded solutions with endpoint impulses (cf. Ferguson (1975) and the Appendix to this paper).

The outer expansion (7) must satisfy the full system (1) within (0,1) as a power series in ε. Thus, the limiting solution, (X_0,Y_0), will satisfy the nonlinear and nonstiff reduced system

$$\dot{X}_0 = f(X_0,Y_0,t,0) , \quad 0 = g(X_0,Y_0,t,0) \tag{9}$$

there. Because g_y is nonsingular, the implicit function theorem guarantees a locally unique solution

$$Y_0(t) = G(X_0,t)$$

of the latter algebraic system, so there remains an m-th order nonlinear system

$$\dot{X}_0 = F(X_0,t) \equiv f(X_0,G(X_0,t),t,0) \tag{11}$$

for X_0. Later terms of the expansion (7) will satisfy linearized versions of the reduced system. The coefficients of ε provide that

$$\dot{X}_1 = f_x(X_0,Y_0,t,0)X_1 + f_y(X_0,Y_0,t,0)Y_1 + f_\varepsilon(X_0,Y_0,t,0) ,$$

and

$$\dot{Y}_0 = g_x(X_0,Y_0,t,0)X_1 + g_y(X_0,Y_0,t,0)Y_1 + g_\varepsilon(X_0,Y_0,t,0) ,$$

so $Y_1(t) = G_xX_1 + g_y^{-1}(\frac{dG}{dt} - g_\varepsilon)$ and $\dot{X}_1 = F_xX_1 + f_yg_y^{-1}(\frac{dG}{dt} - g_\varepsilon) + f_\varepsilon$. More generally, for each $k \ge 1$, we'll obtain a system of the form

$$Y_k(t) = G_x(X_0,t)X_k(t) + \alpha_{k-1}(X_0,\dots,X_{k-1},t) ,$$
$$\dot{X}_k(t) = F_x(X_0,t)X_k(t) + \beta_{k-1}(X_0,\dots,X_{k-1},t) , \tag{12}$$

with successively determined nonhomogeneous terms.

In order to completely specify the outer expansion (7), we must provide boundary conditions for the m vectors $X_k(t)$, $k \geq 0$. Most critically, we first need to provide m boundary conditions for the "slow" vector $X_0(t) = x(t,0)$ in order to determine the limiting solution (X_0,Y_0) within (0,1). It may be natural to attempt to determine them by somehow selecting some subset of m combinations of the m + n boundary conditions (2) and (3) evaluated at $\varepsilon = 0$ (cf. Flaherty and O'Malley (1980) where this is done for certain quasilinear problems). For scalar linear differential equations of higher order, the first such cancellation law was obtained in Wasow's NYU thesis (cf. Wasow (1941, 1944)). For linear systems (with coupled boundary conditions), a (necessarily) more complicated cancellation law is contained in Harris' postdoctoral efforts (cf. Harris (1960, 1973)). These significant early works suggest that we should seek a cancellation law which ignores an appropriate combination of k initial conditions and of $n-k$ terminal conditions, so the limiting solution is determined by a nonlinear m-th order reduced boundary value problem

$$\dot{X}_0 = F(X_0,t) , \qquad 0 \leq t \leq 1$$
$$\Phi(X_0(0)) = 0 , \qquad \Psi(X_0(1)) = 0 , \tag{13}$$

involving $q - k$ initial conditions and $r - n + k$ terminal conditions. Hoppensteadt (1971) considered the reverse problem: Given some solution of a reduced problem, what conditions guarantee that it provides a limiting solution to the original problem (1)-(3) within (0,1). Hadlock (1973) and Freedman and Kaplan (1976) also consider singular perturbations of a given reduced solution, as do Sacker and Sell (1979) who allow the Jacobian g_y to be singular but subject to a "three-band" condition. Vasil'eva and Butuzov (1973) consider problems with special boundary conditions, though their results are extended to more general boundary conditions by Esipova (1975). We note that the numerical solution of a reduced problem like

(13) is much simpler than that of the original problem, because (13) is not stiff and its order is n instead of n + m. Thus, the solution of (13) might be used as an approximate solution of the original (full) problem from which to obtain better approximations by adding boundary layers and by using Newton's method.

Near $t = 0$, the terminal boundary layer correction is negligible, so the representation (4) of our asymptotic solutions requires the initial layer correction $(\varepsilon\xi,\mu)$ to satisfy the nonlinear system

$$\begin{aligned}\frac{d\xi}{d\tau} &= \frac{dx}{dt} - \frac{dX}{dt} = f(X + \varepsilon\xi, Y + \mu, \varepsilon\tau, \varepsilon) - f(X,Y,\varepsilon\tau,\varepsilon) \ , \\ \frac{d\mu}{d\tau} &= \varepsilon\left(\frac{dy}{dt} - \frac{dY}{dt}\right) = g(X + \varepsilon\xi, Y + \mu, \varepsilon\tau, \varepsilon) - g(X,Y,\varepsilon\tau,\varepsilon) \ ,\end{aligned} \tag{14}$$

on $\tau \geq 0$ and to decay to zero as $\tau \to \infty$. This, in turn, provides successive differential equations for the coefficients in the asymptotic expansion

$$\begin{pmatrix}\xi(\tau,\varepsilon)\\ \mu(\tau,\varepsilon)\end{pmatrix} \sim \sum_{j=0}^{\infty}\begin{pmatrix}\xi_j(\tau)\\ \mu_j(\tau)\end{pmatrix}\varepsilon^j \ . \tag{15}$$

Thus, when $\varepsilon = 0$, we have the limiting initial layer system

$$\frac{d\xi_0}{d\tau} = f(X_0(0), Y_0(0) + \mu_0, 0, 0) - f(X_0(0), Y_0(0), 0, 0) \ ,$$

$$\frac{d\mu_0}{d\tau} = g(X_0(0), Y_0(0) + \mu_0, 0, 0) - g(X_0(0), Y_0(0), 0, 0) \ .$$

The decay requirement determines

$$\xi_0(\tau) = -\int_{\tau}^{\infty} \frac{d\xi_0}{d\tau}(s)ds \tag{16}$$

as a functional of μ_0, while μ_0 satisfies a conditionally stable nonlinear system

$$\frac{d\mu_0}{d\tau} = g(X_0(0), Y_0(0) + \mu_0, 0, 0) \equiv \tilde{G}_0(\mu_0; x(0,0))\mu_0 \ . \tag{17}$$

Our hyperbolicity assumption on the eigenvalues of g_y therefore implies that the limiting boundary layer correction is determined by a classical conditional stability problem (17)

on $\tau \geq 0$. The standard theory (cf. Coddington and Levinson (1955) or Hartman (1964) or, in more geometrical terms, Fenichel (1979) and Hirsch et al. (1977)) shows that for each $x(0,0)$, there is (at least locally) a k-manifold $I(x(0,0))$ nontrivially intersecting a neighborhood of the origin such that for

$$\mu_0(0) \in I(x(0,0)) \tag{18}$$

the initial value problem for (17) has a unique solution on $\tau \geq 0$ which decays to zero exponentially as $\tau \to \infty$. One very difficult problem is how to compute the stable initial manifold I, even when $x(0,0)$ is known. Hassard (1979) has begun to address this problem through a Taylor's series approach and Kelley's representation of such stable manifolds through the center manifold theorem.

Recalling that the q limiting initial conditions take the form.

$$A(X_0(0), G(X_0(0),0) + \mu_0(0), 0) = 0 \tag{19}$$

(cf. (2) and (8)), we will assume that it is possible to solve k of these q equations (perhaps nonuniquely) for an isolated solution

$$\mu_0(0) \equiv \gamma(X_0(0)) \in I(X_0(0)) \; . \tag{20}$$

Phrased somewhat differently, in the style of Vasil'eva (1963), we are asking that the initial vector $\mu_0(0)$ for the leading term of this initial layer correction belong to the "domain of influence" of the equilibrium point $\mu_0(\tau) = 0$ of the initial layer system (17) which is itself parameterized by $x(0,0) = X_0(0)$. Rewriting the remaining q - k initial conditions as

$$\Phi(X_0(0)) \equiv \{A(X_0(0), G(X_0(0),0) + \gamma(X_0(0)), 0)\}' = 0 \tag{21}$$

(where the prime indicates the appropriate q - k dimensional subvector), we thereby specify the initial conditions needed for the reduced boundary value problem (13). Because (21) generally depends on γ, we note that the conditions (21) do not simply correspond to a subset of the original initial conditions evaluated along the limiting solution (X_0, Y_0).

In the important quasilinear case when $g_y(x,y,t,0) = \tilde{g}(x,t)$ is independent of y (at least near t = 0), the

resulting initial layer system (17) is linear and $\mu_0(\tau) = e^{G_0\tau}\mu_0(0)$ for $G_0 = \tilde{G}_0(X_0(0),0)$. Thus, μ_0 will decay to zero as $\tau \to \infty$ if $\mu_0(0) = P_0(X_0(0))\mu_0(0)$ where $P_0 = P_0^2$ projects onto the k dimensional stable eigenspace of G_0. (More generally, the manifold I will not coincide with the stable eigenspace of $\tilde{G}_0$.) When we further assume that $A_y(X_0(0),y,0) \equiv \tilde{A}(X_0(0))$ is independent of y, there will be a unique solution $\mu_0(0)$ in the then fixed manifold $I(X_0(0))$, provided the matrix $\tilde{A}(X_0(0)) \cdot P_0(X_0(0))$ has full rank k (cf. Flaherty and O'Malley (1980)).

A simple nonlinear example occurs when y is a scalar and $g(x,y,t,0) = g_1(x,t)y^2 + g_2(x,t)y + g_3(x,t)$. Then μ_0 will satisfy a Riccati equation with solution

$$\mu_0(\tau) = H_0\mu_0(0)/[(H_0 + G_0\mu_0(0))e^{-H_0\tau} - G_0\mu_0(0)]$$

for $G_0 = g_1(X_0(0),0)$ and $H_0 = g_y(X_0(0),Y_0(0),0,0)$ as long as the denominator is nonzero. If $H_0 \geq 0$, only the trivial initial layer correction $\mu_0(\tau) \equiv 0$ will be zero at infinity. With the stability assumption $H_0 < 0$, however, existence on $\tau \geq 0$ and exponential decay at infinity is guaranteed provided $H_0 + G_0\mu_0(0) < 0$. Thus, the magnitude of the initial layer jump $\mu_0(0)$ must be restricted when $G_0\mu_0(0) > 0$.

The terminal layer correction $(\varepsilon\eta(\sigma,\varepsilon),\nu(\sigma,\varepsilon))$ can be analyzed quite analogously to the initial layer correction. In particular, the leading terms (η_0,ν_0) will be determined through exponentially decaying solutions of the conditionally stable terminal layer system

$$\frac{d\nu_0}{d\sigma} = -g(X_0(1),Y_0(1) + \nu_0,1,0) \equiv -\tilde{G}_1(\nu_0;x(1,0))\nu_0 \quad (22)$$

on $\sigma \geq 0$, which has an $n - k$ dimensional manifold $T(x(1,0))$ of initial values $\nu_0(0)$ providing decaying solutions to (22) as $\tau \to \infty$. If we then assume that $n - k$ of the r limiting terminal conditions

$$B(X_0(1),G(X_0(1),1) + \nu_0(0),0) = 0 \quad (23)$$

provide an isolated solution

$$\nu_0(0) \equiv \delta(X_0(1)) \in T(X_0(1)) , \quad (24)$$

the remaining $r - n + k$ conditions provide the terminal conditions

$$\Psi(X_0(1)) \equiv \{B(X_0(1),G(X_0(1),1) + \delta(X_0(1)),0\}'' = 0 \qquad (25)$$

for a reduced boundary value problem (13).

The reduced two-point boundary value problem (13) consists of the nonlinear reduced equation (11) of order m together with the m separated nonlinear boundary conditions (21) and (25). If it is solvable, such a reduced problem can have many solutions. Corresponding to any of its isolated solutions $X_0(t)$, one can expect to obtain a solution of the original problem (1)-(3), for ϵ sufficiently small, which converges to $(X_0, G(X_0,t))$ within $(0,1)$ as $\epsilon \to 0$. Sufficient hypotheses on the corresponding linearized problem to obtain a uniform asymptotic expansion (4) are provided by Hoppensteadt (1971) and others. For this reason, we shall merely indicate the considerations involved in obtaining further terms of the initial boundary layer correction and boundary conditions for higher order terms of the outer expansion.

Further terms of the initial layer correction (15) are determined from the corresponding coefficients of ϵ^k in the nonlinear system (14). Thus, we must have

$$\begin{aligned} \frac{d\xi_k}{d\tau} &= f_y(X_0(0), Y_0(0) + \mu_0(\tau), 0, 0)\mu_k + p_{k-1}(\tau) \ , \\ \frac{d\mu_k}{d\tau} &= g_y(X_0(0), Y_0(0) + \mu_0(\tau), 0, 0)\mu_k + q_{k-1}(\tau) \ , \end{aligned} \qquad (26)$$

for $k \geq 1$, where the nonhomogeneous terms will be exponentially decaying as $\tau \to \infty$ because the preceding ξ_j's, μ_j's, and their derivatives so behave. The homogeneous systems are linearizations of that for (ξ_0, μ_0), and the decaying vector ξ_k will be uniquely provided in terms of μ_k by

$$\xi_k(\tau) = -\int_\tau^\infty \frac{d\xi_k}{d\tau}(s)ds \ . \qquad (27)$$

To obtain μ_k, it is natural to first consider the variable coefficient homogeneous system

$$\frac{d\mu}{d\tau} = \tilde{g}(\tau)\mu \qquad (28)$$

with $\tilde{g}(\tau) = g_y(X_0(0), Y_0(0) + \mu_0(\tau), 0, 0)$. Our hyperbolicity assumption (more specifically, the eigenvalue split for $\tilde{g}(\infty)$

and the exponential convergence of $\tilde{g}(\tau)$ to $\tilde{g}(\infty)$) guarantees that (28) will have k linearly independent exponentially decaying solutions as $\tau \to \infty$. We assume that the split is maintained for all $\tau \geq 0$. If we let $U(\tau)$ be a fundamental matrix for (28) with $U(0) = I$ and let $\tilde{P}_0$ be a constant $n \times n$ matrix of rank k such that $U(\tau)\tilde{P}_0$ provides the linear subspace of decaying solutions to (28), the decaying solution μ_k of (26) must be of the form

$$\mu_k(\tau) = U(\tau)\tilde{P}_0 c_k + \mu_{k-1}^p(\tau) \tag{29}$$

with the particular solution

$$\mu_{k-1}^p(\tau) = \int_0^\tau U(\tau)\tilde{P}_0 U^{-1}(r) q_{k-1}(r)\,dr - \int_\tau^\infty U(\tau)(I - \tilde{P}_0)U^{-1}(r) q_{k-1}(r)\,dr .$$

The vector c_k remains to be determined. In problems where $\tilde{G}_0$ is constant, $\tilde{P}_0$ is the P_0 used for the quasilinear problem. The use of such exponential dichotomies (cf. Coppel (1978)) in the singular perturbations context goes back to Levin and Levinson (1954). Indeed, the "roughness" of the exponential dichotomy might be used to justify the use of $U(\tau)\tilde{P}_0$ all the way back to $\tau = 0$. Proceeding analogously, the terminal layer term $\nu_k(\sigma)$ will be determined in the form

$$\nu_k(\sigma) = V(\sigma)\tilde{P}_1 d_k + \nu_{k-1}^p(\sigma) , \tag{30}$$

where $V(\sigma)\tilde{P}_1$ is assumed to span an $n - k$ dimensional space of decaying solutions to

$$\frac{d\nu}{d\sigma} = -g_y(X_0(1), Y_0(1) + \nu_0(\sigma), 1, 0)\nu$$

on $\sigma \geq 0$ and $\nu_{k-1}^p(\sigma)$ is exponentially decaying and successively determined. $\eta_k(\sigma)$ will uniquely follow from ν_k by integrating $d\eta_k/d\sigma$. To complete the formal determination of our expansion (4), we must successively specify the constants c_k in (29), d_k in (30), and the m boundary conditions for each $X_k(t)$, $k > 0$.

Since $x(0,\epsilon) \sim X(0,\epsilon) + \epsilon\xi(0,\epsilon)$ and $y(0,\epsilon) \sim Y(0,\epsilon) + \mu(0,\epsilon)$, the coefficient of ϵ^k (for any $k > 0$) in the initial condition (2) implies that

$$A_x(X_0(0),Y_0(0) + \mu_0(0),0)X_k(0) + A_y(X_0(0),Y_0(0) + \mu_0(0),0)(Y_k(0) + \mu_k(0))$$

is successively determined. Since $Y_k(0) - G_x(X_0(0),0)X_k(0)$ and $\mu_k(0) - \tilde{P}_0 c_k$ are also known in terms of preceding coefficients, we have

$$(A_{x0} + A_{y0}G_{x0})X_k(0) + A_{y0}\tilde{P}_0 c_k = \delta_{k-1} \tag{31}$$

determined termwise. (The zero subscripts indicate evaluation along $(x(0,0),y(0,0),0)$.) Assuming that the matrix

$$A_y(x(0,0),y(0,0),0)\tilde{P}_0 \tag{32}$$

has its maximal rank k, it will be possible to (perhaps nonuniquely) solve k of the q equations (31) for

$$c_k = \tilde{P}_0 c_k = (A_{y0}\tilde{P}_0)^{\dagger}[\delta_{k-1} - (A_{x0} + A_{y0}G_{x0})X_k(0)] \tag{33}$$

(with the dagger representing the matrix pseudoinverse). This leaves the remaining $q - k$ initial conditions

$$\tilde{\gamma}X_k(0) = \xi_{k-1} \tag{34}$$

to be solved for $X_k(0)$. Here

$$\tilde{\gamma} = [I - A_{y0}\tilde{P}_0(A_{y0}\tilde{P}_0)^{\dagger}](A_{x0} + A_{y0}G_{x0}) ,$$

$\xi_{k-1} = A_{y0}\tilde{P}_0(A_{y0}\tilde{P}_0)^{\dagger}\delta_{k-1}$, and $\tilde{\gamma}$ will have rank $q - k$.

In analogous fashion, if the matrix

$$B_y(x(1,0),y(1,0),0)\tilde{P}_1 \tag{35}$$

has its maximal rank $n - k$, we can use $n - k$ of the terminal conditions (3) to (generally nonuniquely) provide $d_k = \tilde{P}_1 d_k$ in (30), and the remaining $r - n + k$ terminal conditions

$$\tilde{\delta}X_k(1) = \eta_{k-1} \tag{36}$$

for the outer expansion term $X_k(t)$. Here, η_{k-1} is known successively and

$$\tilde{\delta} = [I - B_{y1}\tilde{P}_1(B_{y1}\tilde{P}_1)^{+}](B_{x1} + B_{y1}G_{x1})$$

with the subscript 1 indicating evaluation at $(x(1,0),y(1,0),1)$.

Putting everything together, we've shown that the k-th term in the outer expansion should satisfy an m-th order linear boundary value problem

$$\begin{aligned} \dot{X}_k &= F_x(X_0,t)X_k + \beta_{k-1}(X_0,\ldots,X_{k-1},t) , \\ \tilde{\gamma}X_k(0) &= \xi_{k-1} , \qquad \tilde{\delta}X_k(1) = \eta_{k-1} , \end{aligned} \tag{37}$$

with successively determined nonhomogeneities β_{k-1}, ξ_{k-1}, and η_{k-1}. These problems for all $k > 0$ will have unique solutions $X_k(t)$ provided the corresponding homogeneous system

$$\dot{X} = F_x(X_0,t)X ,$$

$$\tilde{\gamma}X(0) = 0 , \qquad \tilde{\delta}X(1) = 0 ,$$

has only the trivial solution on $0 \le t \le 1$. Otherwise, one must impose appropriate orthogonality conditions on (37) to achieve solvability.

Altogether, we've determined possibilities for constructing multiple formal asymptotic solutions to our boundary value problem (1)-(3) in the form (4). To prove that the corresponding solutions exist *for ϵ sufficiently small* requires some further analysis (cf. Hoppensteadt (1971), Vasil'eva and Butuzov (1973), and Eckhaus (1979)), but no surprises.

3. A SIMPLE EXAMPLE.

A relatively simple example is provided by the harmonic oscillator system

$$\begin{aligned} \dot{x} &= 1 - x , \\ \epsilon\dot{y}_1 &= y_2 , \\ \epsilon\dot{y}_2 &= \alpha^2(x)y_1 + \beta(x) , \end{aligned} \tag{38}$$

where

$$\alpha(x) = 1 + 2x \qquad \text{and} \qquad \beta(x) = 8x(1 - x) .$$

Since $m = k = n - k = 1$ whenever $\alpha(x)$ remains nonzero, we can expect one dimensional boundary layer behavior at each endpoint. Using the three linear boundary conditions

$$\begin{aligned} &x(0) + y_1(0) = 0 , \\ &-bx(0) + y_2(0) = 0 , \\ &x(1) + y_1(1) = 0 , \end{aligned} \tag{39}$$

we could hope that the limiting interior behavior is determined by the reduced system

$$\begin{aligned} &\dot{X}_0 = 1 - X_0 , \\ &Y_{20} = 0 , \\ &\alpha^2(X_0)Y_{10} + \beta(X_0) = 0 \end{aligned} \tag{40}$$

and a combination of the limiting initial forms $X_0(0) + Y_{10}(0)$ and $-bX_0(0)$ set to zero.

By expliciting analyzing the initial layer system and determining the appropriate projection matrix P_0, we find the initial condition $\Phi(X_0(0)) = 0$ (needed for the reduced solution) to be

$$|\alpha(X_0(0))|(X_0(0) + Y_{10}(0)) - bX_0(0) = 0 . \tag{41}$$

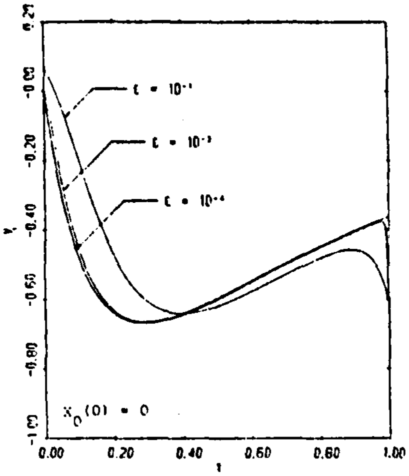
Rewriting this as a cubic polynomial, we obtain the three roots

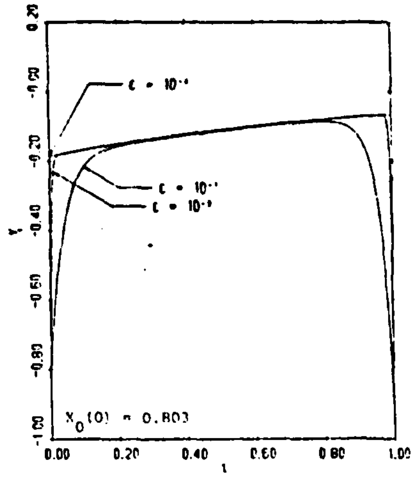
$$X_0(0) = 0$$

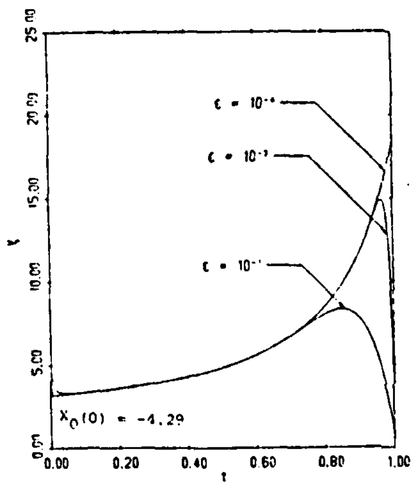
and

$$X_0(0) = -\frac{3}{2} + \frac{1}{4} \operatorname{sgn} \alpha_0\left[b + \sqrt{(b \operatorname{sgn} \alpha_0 - 4)^2 + 48}\right]$$

arranged so that $\operatorname{sgn} \alpha_0 = \pm 1$ depending on the sign of $1 + 2X_0(0)$. All roots are appropriate for determining an X_0 except for that range of initial values for which the resulting $\alpha(X_0(t)) = 3 + 2(X_0(0) - 1)e^{-t}$ has a zero in $0 \le t \le 1$. Then, the equation for y_2 has a turning point and $Y_{10}(t)$ becomes unbounded. When $bX_0(0) = 0$, both of the initial conditions for the system are satisfied by the limiting solution, and there is no nonuniform convergence at $t = 0$.







FIGURES 1, 2 and 3. The three solutions $y_1(t,\varepsilon)$ to our example for $b = 2$.

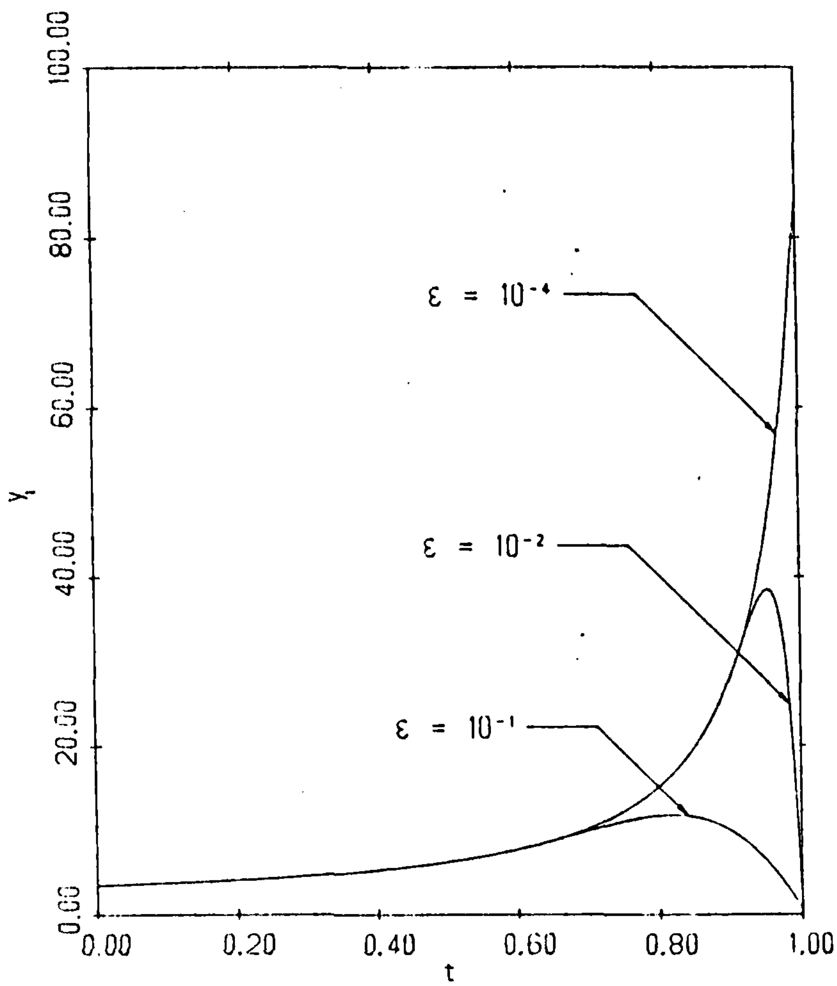


FIGURE 4. The solution $y_1(t,\varepsilon)$ for $b = 0$ and $x_0(0) = -7/2$.

Otherwise, there is an initial layer with a nontrivial $\mu_0(\tau)$. Integrating $X_0(t)$ to $t = 1$ (or evaluating the explicit solution) also determines $Y_{10}(1)$. If $X_0(1) + Y_{10}(1) \neq 0$, the terminal layer correction $\nu_0(\sigma)$ is also nontrivial.

For $b = 2$, we obtain the three roots $X_0(0) = 0$, 0.803, and -4.29 and three corresponding asymptotic solutions. For small values of ε, the asymptotic solution can be used as a first guess to obtain numerical solutions (cf. Flaherty and O'Malley (1980) and Figures 1-3). For $b = 0$ and $X_0(0) = -7/2$, $\alpha(X_0(t))$ has a zero above $t = 1$. This forces the boundary layer jump $|Y_{10}(0) - y(1,0)|$ to be large (89.8 compared to 0.4 and 0.6 for the other $b = 0$ roots, $X_0(0) = 0$ and $1/2$) (cf. Figure 4).

APPENDIX: THE CONSTRUCTION OF IMPULSIVE SOLUTIONS TO QUASILINEAR PROBLEMS.

It is basic to the preceding development that we can solve the limiting boundary conditions for initial values of the boundary layer correction terms (i.e. $\mu_0(0)$ in (19) and $\nu_0(0)$ in (23)). Moreover, the solutions must lie on the stable manifolds I and T for the corresponding boundary layer systems. This will certainly be impossible when the limiting boundary conditions $A(x(0),y(0),0)$ or $B(x(1),y(1),0)$ are independent of y and suggests that the corresponding asymptotic solutions will then no longer have the form (4). Indeed, the earlier work of O'Malley (1970) (with $m = n = 1$) suggests that the endpoint behavior will then be more singular (i.e., impulsive). With the resulting unboundedness in the y vector, we ask for linearity in the fast variables in order to generate expansions.

As a sample problem, consider

$$\begin{aligned} \frac{dx}{dt} &= f_1(x,t,\varepsilon) + f_2(t,\varepsilon)y , \\ \varepsilon \frac{dy}{dt} &= g_1(x,t,\varepsilon) + g_2(t,\varepsilon)y , \end{aligned} \tag{42}$$

where $g_2(t,0)$ has $k > 0$ stable and $n - k > 0$ unstable eigenvalues throughout $0 \leq t \leq 1$, subject to the $q + r = m + n$ boundary conditions

$$A(x(0),y(0),\varepsilon) \equiv A_1(x(0),\varepsilon) + \varepsilon A_2(\varepsilon)y(0) = 0 \ ,$$
$$B(x(1),y(1),\varepsilon) = 0 \ . \tag{43}$$

Since $A_y \equiv 0$ at $\varepsilon = 0$, we shall now seek asymptotic solutions in the form

$$x(t,\varepsilon) = X(t,\varepsilon) + \xi(\tau,\varepsilon) + \varepsilon\eta(\sigma,\varepsilon) \ ,$$
$$y(t,\varepsilon) = Y(t,\varepsilon) + \frac{1}{\varepsilon}\,\mu(\tau,\varepsilon) + \nu(\sigma,\varepsilon) \ , \tag{44}$$

(as an alternative to (4)) where the terms are all expandable in ε and the boundary layer corrections decay to zero as before.

The limiting solution (X_0,Y_0) within $(0,1)$ satisfies the reduced system

$$\frac{dX_0}{dt} = F(X_0,t) \equiv f_1(X_0,t,0) + f_2(t,0)G(X_0,t) \ , \tag{45}$$

where

$$Y_0(t) = G(X_0,t) \equiv -g_2^{-1}(t,0)g_1(X_0,t,0) \ . \tag{46}$$

The initial layer correction $(\xi,\frac{1}{\varepsilon}\,\mu)$ now satisfies the almost linear system

$$\frac{d\xi}{d\tau} = f_2(\varepsilon\tau,\varepsilon)\mu + \varepsilon[f_1(X+\xi,\varepsilon\tau,\varepsilon) - f_1(X,\varepsilon\tau,\varepsilon)] \ ,$$

$$\frac{d\mu}{d\tau} = g_2(\varepsilon\tau,\varepsilon)\mu + \varepsilon[g_1(X+\xi,\varepsilon\tau,\varepsilon) - g_1(X,\varepsilon\tau,\varepsilon)] \ ,$$

so its leading coefficients satisfy the constant linear system

$$\frac{d\xi_0}{d\tau} = F_0\mu_0 \ , \qquad \frac{d\mu_0}{d\tau} = G_0\mu_0 \ ,$$

for $(F_0,G_0) = (f_2(0,0),g_2(0,0))$. The decaying solutions are

$$\mu_0(\tau) = e^{G_0\tau}P_0\mu_0(0) \ ,$$
$$\xi_0(\tau) = F_0G_0^{-1}\mu_0(\tau) \ , \tag{47}$$

where P_0 projects onto the constant k dimensional stable eigenspace I of G_0. These vectors and higher order terms in the initial layer correction will necessarily lie in this same (known) eigenspace. With the expansion (44), our initial condition takes the limiting form

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$$A_1(X_0(0) + F_0G_0^{-1}P_0\mu_0(0), 0) + A_2(0)P_0\mu_0(0) = 0 \ . \qquad (48)$$

We wish to solve k of these q nonlinear equations for

$$P_0\mu_0(0) \equiv \gamma(X_0(0)) \ .$$

The solution will be locally unique if the appropriate $k \times k$ Jacobian is nonzero. The remaining $q - k$ initial conditions will then provide the initial values

$$\begin{aligned} \Phi(X_0(0)) \equiv \{A(X_0(0) + F_0G_0^{-1}\gamma(X_0(0)), 0) \\ + A_2(0)\gamma(X_0(0))\}' = 0 \ , \end{aligned} \qquad (49)$$

for the limiting solution $X_0(t)$.

Preceding analogously, the limiting terminal layer correction will satisfy the linear system

$$\frac{d\eta_0}{d\sigma} = -F_1\nu_0 \ , \qquad \frac{d\nu_0}{d\sigma} = -G_1\nu_0 \ ,$$

where $(F_1, G_1) = (f_2(1,0), g_2(1,0))$. The decaying solution is

$$\begin{aligned} \nu_0(\sigma) &= e^{-G_1\sigma} P_1\nu_0(0) \ , \\ \eta_0(\sigma) &= F_1G_1^{-1}\nu_0(\sigma) \ , \end{aligned} \qquad (50)$$

where matrix P_1 projects onto the constant $n - k$ dimensional unstable eigenspace T of G_1. The limiting terminal conditions take the form

$$B(X_0(1), G(X_0(1), 1) + P_1\nu_0(0), 0) = 0 \ . \qquad (51)$$

Assuming that we can solve (perhaps nonuniquely) $n - k$ of these equations for

$$P_1\nu_0(0) \equiv \delta(X_0(1)) \ , \qquad (52)$$

there will remain $r - n + k$ terminal conditions

$$\Psi(X_0(1)) \equiv \{B(X_0(1), G(X_0(1), 1) + \delta(X_0(1), 0)\}'' = 0 \qquad (53)$$

for X_0, as in (25). Thus, any limit of a solution of the form (44) must be determined by

$$\begin{aligned} \dot{X}_0 &= F(X_0, t) \ , \\ \Phi(X_0(0)) &= 0 \ , \qquad \Psi(X_0(1)) = 0 \ . \end{aligned} \qquad (54)$$

Assuming this reduced problem is solvable, we should be able to construct higher order approximations to solutions without unusual difficulty.

We wish to point out that the initial impulse $\frac{1}{\varepsilon}\mu_0(\tau) = \frac{1}{\varepsilon} e^{G_0 t/\varepsilon} P_0 \mu_0(0)$ in the representation (44) of the fast vector y behaves like a multiple of a matrix delta function in the limit $\varepsilon \to 0$. It forces the rapid transfer (i.e., the non-uniform convergence) of the slow vector x from $x(0,0) = X_0(0) + \xi_0(0) = X_0(0) + F_2 G_0^{-1} P_0 \mu_0(0)$ to $X_0(0)$ in the k dimensional range I of the stable eigenspace of G_0. This is in contrast to the initial Heaviside jump in the y vector and the uniform convergence of the x vector for asymptotic solutions of the form (4).

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Program in Applied Mathematics
University of Arizona
Tucson, AZ 85721

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